large cost savings. One avenue of future work in this area is incorporating the effect of the correlation of the sine loads for harmonics of a single rotating shaft.

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Optimum Thrust-to-Weight Ratio for Gravity-Turn Trajectories

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Nomenclature

a = vehicle thrust-to-weight ratio
g = gravitational acceleration, ms^{-2}
k = velocity constant, ms^{-1}
m = vehicle mass, kg
T = vehicle thrust, N
t = time, s
u = vehicle velocity, ms^{-1}
v_e = exhaust velocity, ms^{-1}
v_e^* = effective exhaust velocity, ms^{-1}
z = $\tan(\phi/2)$
ΔV_g = gravity loss, ms^{-1}
λ = payload ratio
μ_e = engine thrust-to-weight ratio
μ_{es} = design constant
σ_{es} = design constant
ϕ = vehicle flight path angle, rad

Subscripts

b = at burnout
opt = optimum (for maximum λ)
0 = initial

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Introduction

THE analysis of the ascent of a single-stage rocket from a planetary surface to orbit on a gravity-turn trajectory is well known.[1–4] This Note extends the analysis to quickly arrive at an expression for the burnout-to-initial mass ratio of a single-stage vehicle ascending in vacuo and thereafter derives some bounds on the optimum initial vehicle thrust-to-weight ratio. The analysis is not directly useful for detailed studies of vehicle performance because a gravity turn is not an optimal trajectory, but it may still be educationally instructive, or possibly useful for concept evaluation and preliminary design studies.

Analysis of a Single-Stage Rocket on a Gravity Turn

In a flat-Earth approximation,[1,2] ignoring atmospheric drag, the equations of motion of a rocket vehicle ascending on a gravity turn (with no thrust vectoring) are

$$m\frac{\mathrm{d}u}{\mathrm{d}t} = T - mg\cos\phi \tag{1a}$$

$$u\frac{\mathrm{d}\phi}{\mathrm{d}t} = g\sin\phi \tag{1b}$$

where ϕ is the angle of the flight path from the vertical, T is the vehicle thrust, m is the vehicle flight mass, u is the vehicle velocity, and g is the gravitational acceleration (assumed to be constant and equal to the surface value, g_0).

In the analysis that follows, it will be assumed that the vehicle thrust-to-weight ratio a is held constant at the initial (liftoff) value throughout the ascent:

$$a = T/mg = T_0/m_0g_0 = a_0 \tag{2}$$

In this particular case, the substitution $z = \tan(\phi/2)$ yields the solution[1]

$$u = kz^{a-1}(1+z^2) \tag{3a}$$

$$\frac{\mathrm{d}u}{\mathrm{d}z} = k\{(a-1)z^{a-1} + (a+1)z^a\} \tag{3b}$$

where the constant k is equal to half the burnout velocity, $k = u_b/2$, if and when $u = u_b$ and $m = m_b$ at $z = 1$ (a condition that is assumed hereafter).

If the thrust is approximated by

$$T = -v_e\frac{\mathrm{d}m}{\mathrm{d}t} \tag{4}$$

where the exhaust velocity v_e is assumed to be constant, then the rocket's acceleration is given by

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{v_e^*}{m}\frac{\mathrm{d}m}{\mathrm{d}t} \tag{5}$$

where v_e^* is an effective exhaust velocity, $v_e^* = v_e(1 - a^{-1}\cos\phi)$. The burnout-to-initial mass ratio m_b/m_0 is found by integration with respect to velocity from $u = 0$ to the burnout velocity $u = u_b$:

$$\ln\left\{\frac{m_b}{m_0}\right\} = -\int_0^{u_b}\frac{\mathrm{d}u}{v_e^*} \tag{6}$$

Noting $\cos\phi = (1-z^2)/(1+z^2)$, this integral becomes

$$\ln\left\{\frac{m_b}{m_0}\right\} = -\int_0^{u_b}\left\{1 - \frac{(1-z^2)}{a(1+z^2)}\right\}^{-1}\frac{\mathrm{d}u}{v_e}$$

$$= -\frac{u_b}{v_e}\int_0^1 \frac{1}{2}a(1+z^2)z^{a-2}\,\mathrm{d}z \tag{7}$$

which can easily be solved to yield a simple expression:

$$\frac{m_b}{m_0} = \exp\left\{-\frac{u_b}{v_e}\frac{a^2}{(a^2-1)}\right\} \tag{8}$$

Hence, it can be seen that the burnout-to-initial mass ratio is lower than the ideal value, $\exp\{-u_b/v_e\}$, and the gravity loss incurred during the ascent is[2]

$$\Delta V_g = v_e \ln\{m_0/m_b\} - u_b = u_b/(a^2 - 1) \tag{9}$$

Optimum Thrust-to-Weight Ratio

Suppose that in a conceptual design study the initial mass m_0 of the vehicle considered is fixed, and the vehicle mass breakdown is written as

$$m_0 = m_b + m_{\text{propellant}} \tag{10a}$$

$$m_b = m_{\text{payload}} + m_{\text{misc}} + m_{\text{structure}} + m_{\text{engine}} \tag{10b}$$

where $m_{\text{propellant}}$ is the ascent propellant mass, m_{payload} is the payload, $m_{\text{structure}}$ is the structural mass, m_{engine} is the total engine mass, and m_{misc} is the mass of all other subsystems and miscellaneous items. Hence, for a prescribed value of v_e/v_b, the payload-to-initial mass ratio, $\lambda = m_{\text{payload}}/m_0$, is maximized when $m_b - m_{\text{misc}} - m_{\text{structure}} - m_{\text{engine}}$ is maximized.

Assume to begin with that m_{misc} and $m_{\text{structure}}$ are fixed. In this case λ is maximized when $m_b - m_{\text{engine}}$ is maximized. If the engine thrust-to-weight ratio $\mu_e = T_0/(g_0 m_{\text{engine}})$ is held constant, then [differentiating Eq. (8) with respect to a] it follows that λ is a maximum when $a = a_{\text{opt}}$ given by

$$\frac{2a_{\text{opt}}}{\left(a_{\text{opt}}^2 - 1\right)^2} = \mu_e^{-1}\frac{v_e}{u_b}\frac{m_0}{m_b} = \mu_e^{-1}\frac{v_e}{v_b}\exp\left\{\frac{u_b}{v_e}\frac{a_{\text{opt}}^2}{\left(a_{\text{opt}}^2 - 1\right)}\right\} \tag{11}$$

Now assume that m_{misc} and $m_{\text{structure}}$ vary as the vehicle design thrust-to-weight ratio is altered. An exact expression can no longer be defined (because the vehicle's structure and subsystem masses will depend on a variety of design factors[5]), but to first order it is not unreasonable to adopt an approximate linear relation:

$$(m_{\text{structure}} + m_{\text{engine}} + m_{\text{misc}})/m_0 = \sigma_{es} + \mu_{es}^{-1}a \tag{12}$$

where σ_{es} and μ_{es} are constants (and $\mu_{es} < \mu_e$). In this more general case, the condition for maximum λ is the same as Eq. (11) except the design constant μ_{es} replaces μ_e:

$$\frac{2a_{\text{opt}}}{\left(a_{\text{opt}}^2 - 1\right)^2} = \mu_{es}^{-1}\frac{v_e}{u_b}\frac{m_0}{m_b} \tag{13}$$

The optimum value of initial thrust-to-weight implied by Eq. (13) has a lower limit. The minimum feasible value of a_{opt} occurs as $m_s + m_e + m_{\text{misc}} \to m_b$ (or as $\lambda \to 0$) and also as $\sigma_{es} \to 0$, that is, when

$$\frac{2a_{\text{opt}}}{\left(a_{\text{opt}}^2 - 1\right)^2} \to \frac{v_e}{u_b}\frac{\mu_{es}^{-1}}{\left(\mu_{es}^{-1}a_{\text{opt}} + \sigma_{es}\right)} \to a_{\text{opt}}^{-1}\frac{v_e}{u_b} \tag{14}$$

or (rearranging this equation and solving the resulting quadratic) when

$$a_{\text{opt}} \to a_{\text{opt}_{\min}} = \left(\tfrac{1}{2}u_b/v_e\right)^{\frac{1}{2}} + \left(1 + \tfrac{1}{2}u_b/v_e\right)^{\frac{1}{2}} \tag{15}$$

Hence, irrespective of the actual values of σ_{es} and μ_{es}, the optimum vehicle thrust-to-weight ratio a_{opt} lies within certain bounds given by Eqs. (11) and (15), provided positive λ is achievable and provided λ needs to be maximized.

Limitations of the Analysis

Note that, in reality, constant in-flight vehicle thrust-to-weight ratio is unlikely. If a vehicle has multiple engines, then these engines might be shut down sequentially during the ascent to prevent acceleration limits from being exceeded; but it is unlikely that each engine would be throttled in such a way that the total thrust varies directly in proportion to vehicle flight mass. Moreover gravity-turn trajectories (with/without the condition assumed herein that burnout occurs at $z = 1$) are not optimal.[4] Nevertheless, despite these differences, the optimum conditions for vehicle thrust-to-weight found here appear to be useful for concept evaluation and preliminary estimates. To demonstrate this assertion, consider the following examples.

Single-Stage Rocket Ascent from Earth's Surface to Orbit

As a first example, consider a single-stage-to-orbit launch vehicle ascending to low Earth orbit, using liquid hydrogen and oxygen propellants, such that $v_e \cong 4500\ \text{ms}^{-1}$ and $v_e/u_b = 0.5$. If $\mu_e = 50$ (a typical value with current technology), then $m_b - m_{\text{engine}}$ is maximized when $m_b/m_0 \cong 0.1$ and $a_{\text{opt}} \cong 3$. If structural mass increases significantly with launch acceleration, then the optimum initial vehicle thrust-to-weight ratio (for maximum payload ratio) will be lower, but it will not be less than $a_{\text{opt}_{\min}} = 1 + \sqrt{2}$ at which point $m_b/m_0 \cong 0.089$. Hence, λ is maximized when a_{opt} lies between approximately 2.4 and 3. Note, however, that in reality this result ignores the effects of drag losses, etc. Hence, a lower initial vehicle thrust-to-weight ratio might be expected.

Apollo Lunar Module Ascent

As a second example, consider the ascent stage of the Apollo Lunar Module,[5] which had a near-constant ascent thrust of about $T = 15.57$ kN and an initial mass of about $m_0 = 4700$ kg on the lunar surface where $g_0 \cong 1.62\ \text{m/s}^2$, such that $a_0 \cong 2$ (depending more exactly on the mission number: Apollo 11–17). To attain lunar orbital velocity at $u_b \cong 1.63$ km/s, with an exhaust velocity of about $v_e = 3050\ \text{ms}^{-1}$, Eq. (8) gives $m_b/m_0 \cong 0.495$, which is slightly worse than the actual value of 0.52. Furthermore, assuming $m_e = 213$ kg (the mass of the ascent propulsion system[5]) such that $\mu_e \cong 45$, Eq. (11) gives $a_{\text{opt}} \cong 3.1$, and Eq. (15) gives $a_{\text{opt}_{\min}} \cong 1.64$. Hence, despite the differences in trajectory thrust profile, etc., the simple analysis presented herein does not give unreasonable results.

Acknowledgments

The work presented in this Note was initiated in 1988, while I was attending the International Space University (ISU) summer session at the Massachusetts Institute of Technology, Boston. It was first submitted to this journal on 20 July 1999, exactly 30 years after the Apollo 11 moon landing, as a deliberate tribute to that event. With that background in mind, I would like to thank the founders of ISU and the participants of the 1988 summer session, in particular Edwin E. ("Buzz") Aldrin, for his encouragement and for stressing the importance of basic trajectory/orbital analysis, even though this work is just a small contribution and somewhat overdue.

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